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MICROCOPY RESOLUTION TEST CHART

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS
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1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
AFOSR-TR-79-1189		

4. TITLE (and Subtitle)

A FUNCTIONAL EXPANSION APPROACH TO THE
SOLUTION OF NONLINEAR FEEDBACK PROBLEMS.

5. TYPE OF REPORT & PERIOD COVERED

Interim *rept.*

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)

Ram-Nandan P. Singh
Timothy L. Johnson

8. CONTRACT OR GRANT NUMBER(s)

AFOSR-77-3281

9. PERFORMING ORGANIZATION NAME AND ADDRESS

M.I.T.
Laboratory for Information & Decision Systems
Cambridge, MA 02139

10. PROGRAM ELEMENT, PROJECT, TASK
AREA & WORK UNIT NUMBERS

61102F 2304/A1

11. CONTROLLING OFFICE NAME AND ADDRESS

Air Force Office of Scientific Research/NM
Bolling AFB, Washington, DC 20332

12. REPORT DATE

August 1979

13. NUMBER OF PAGES

23

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)

15. SECURITY CLASS. (of this report)

UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING
SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

LIDS-P-796-A

DDC

RECEIVED

NOV 29 1979

A

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

THIS PAPER HAS BEEN RE-SUBMITTED TO THE IEEE TRANSACTIONS ON AUTOMATIC CONTROL

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Hamilton-Jacobi-Bellman Equation, Continuation Methods, Functional Expansion
Techniques, Nonlinear Control

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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associated "linear-quadratic" optimal control problem.

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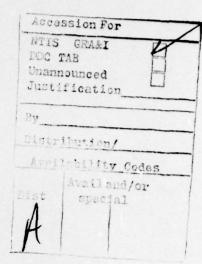
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A FUNCTIONAL EXPANSION APPROACH TO THE SOLUTION
OF NONLINEAR FEEDBACK PROBLEMS*

by

Ram-Nandan P. Singh, Member IEEE**
and
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ABSTRACT

The application of a functional expansion technique introduced by
Bellman to the determination of nonlinear optimal control laws is described.
For a certain class of "smooth" control problems, it is demonstrated that
convergence of the expansion technique may be improved by use of a non-
linear (rather than linear) auxiliary equation which is based on a known
cost-to-go function for an associated "linear-quadratic" optimal control
problem.

This paper has been re-submitted to the <u>IEEE Transactions on Automatic Control.</u>

79 11 27 168

I. Introduction

Under certain smoothness and controllability assumptions the solutions
of broad class of optimal control problems is described by the Hamilton-
Jacobi-Bellman equation [1]. Under some further restrictions, the optimal
state-feedback control law can be computed analytically once the solution
of this equation is known. In certain applications (e.g., [2]), the
global nonlinear control law has been shown to offer significant advantages,
in performance as well as implementation, over local linearized control laws.

The problem of obtaining approximate solutions of the Hamilton-Jacobi-
Bellman equations, therefore, has received considerable attention. Quasi-
linearization [3], power series [4], and global analysis techniques [5],
[6], [7], to mention but a few examples, have been applied to this problem.
The purpose of this brief paper is to point out a functional expansion
technique which is applicable to the solution of the nonlinear partial
differential equation that is obtained when the optimal control can be
expressed analytically in terms of the cost-to-go. Analytical examples
are presented which provide an indication of the nature of convergence of
the method. A complete numerical analysis of convergence is not provided
here (and is probably best pursued in the context of individual applications);
but it is suggested that the functional expansion approach is often better-
suited than power series methods for numerical computations.

II.   Nonlinear State-Feedback Laws for a Class of Optimal Control Problems

In this section, we give an explicit statement of the class of optimal control problems which can be addressed using the functional expansion techniques of the following section, and display the Hamilton-Jacobi-Bellman (HJB) equation characterizing the optimal control law.

Let $[t_0, t_1]$ denote the time interval of the problem and let the space $\mathcal{U}$, of admisable controls be a convex open subset of $PC([t_0, t_1]; R^r)$, the space of piecewise continuous functions on $[t_0, t_1]$ taking values in R^r. We seek an optimal control $u* \varepsilon \mathcal{U}$ to minimize the functional $J: \mathcal{U} \to R$ defined by

$$J(u) = \int_{t_0}^{t_1} L(x(t), u(t)) dt \tag{1}$$

subject to the constraints

$$\dot{x}(t) = f(x(t), u(t)) \tag{2}$$

$$x(t_0) = x_0 \tag{3}$$

The functional $L: R^n x R^r \to R$ is assumed to take the form

$$L(x, u) = u' u + Q(x) \tag{4}$$

where $Q(x)$ satisfies

(i) $Q(x) = Q(-x)$, and $Q(x) > 0$ for all $x \varepsilon R^n$, $x \neq 0$.

(ii) $\lim_{||x|| \to 0} Q(x) = 0$; $\lim_{||x|| \to \infty} \frac{Q(x)}{||x||} \geq q > 0$

(iii) $Q(x)$ is analytic in x.

The function $f: R^n x R^r \to R^n$ is assumed to take the form

$$f(x,u) = Ax + Bu + \tilde{f}(x,u) \tag{5}$$

where

$$\tilde{f}(x,u) = f_0(x) + \sum_{i=1}^{n} e_i u'(F_{1i}f_1(x) + F_{2i}(x)u) \tag{6}$$

The following additional assumptions are imposed on f:

(iv) (A,B) is a completely controllable pair.

(v) $\tilde{f}(x,u)$ is analytic in x, for all $u \in R^r$.

(vi) $\lim\limits_{\substack{||x|| \to 0 \\ ||u|| \to 0}} \dfrac{||\tilde{f}(x,u)||}{(||x|| + ||u||)} = 0$

(vii) $\lim\limits_{||x|| \to 0} ||f_0(x)|| = 0; \quad \lim\limits_{||x|| \to 0} ||F_{2i}(x)|| = 0.$

In (6), e_i denotes the ith unit basis vector, $f_0: R^r \to R^n$, $f_1: R^n \to R^n$, $F_{1i} \in R^{rxn}$, and $F_{2i}: R^n \to R^{rxr}$. Under assumptions (iv)-(vii), it can be shown [8] that (2) is (uniformly) completely controllable.[1]

Consider the truncated problem obtained by replacing t_0 by $\tau \in [t_0, t_1]$ and x_0 by x_τ in (1)-(3) and assume that there exists an optimal control u_τ^* for each τ. Let the value function

$$V(x_\tau, \tau): R^n x [t_0, t_1] \to R \tag{7}$$

be defined as the (minimum) value of the cost-to-go, which is achieved for control u_τ^*. V is assumed to be twice continuously differentiable with respect to x and continuously

---

[1] The time-varying case can also be considered, by requiring that the foregoing properties hold uniformly for $t \in [t_0, t_1]$.

differentiable in t.

Then the value function satisfies the Hamilton-Jacobi-Bellman equation for problem (1)-(3)

$$\frac{\partial V}{\partial t} = - \min_{u(t)} \{u' u + Q(x) + \frac{\partial V'}{\partial x} f(x,u)\} \quad ; \quad V(x,t_1) \equiv 0 \qquad (8)$$

The minimizing control is given in feedback form by

$$u^*(t) = -\frac{1}{2}\{I + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x^*(t),t)[F'_{2i}(x^*(t)) + F_{2i}(x^*(t))]\}^{-1} \cdot$$

$$\{B' \frac{\partial V}{\partial x}(x^*(t),t) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x^*(t),t)F_{1i} f_1(x^*(t))\} \qquad (9)$$

Inserting (9) in (8) and simplifying terms gives rise to a partial differential equation in the cost-to-go, which is written-out here for future reference:

$$-\frac{\partial V}{\partial t} = \frac{1}{4}\left[\frac{\partial V'}{\partial x}B + \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}f_1'F_{1i}'\right]\left(I + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}(F_{2i}'+F_{2i})\right)^{-2} \cdot$$

$$\left[B'\frac{\partial V}{\partial x} + \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}F_{1i}f_1\right] + Q + \frac{\partial V'}{\partial x}(Ax + f_0)$$

$$-\frac{1}{2}\frac{\partial V'}{\partial x}B\left[I + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}(F_{2i}' + F_{2i})\right]^{-1}\left[B'\frac{\partial V}{\partial x} + \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}F_{1i}f_1\right]$$

$$-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial V}{\partial x_j}\left[\frac{\partial V'}{\partial x}B + \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}f_1'F_{1i}'\right] \cdot$$

$$\left[I + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}(F_{2i}' + F_{2i})\right]^{-1}F_{1j}f_1$$

$$+\frac{1}{4}\sum_{j=1}^{n}\frac{\partial V}{\partial x_j}\left[\frac{\partial V'}{\partial x}B + \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}f_1'F_{1i}'\right]\left[I + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}(F_{2i}' + F_{2i})\right]^{-1}$$

$$F_{2j}\left[I + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}(F_{2i}' + F_{2i})\right]^{-1}\left[B'\frac{\partial V}{\partial x} + \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}F_{1i}f_1\right]$$

With the boundary condition $V(x,t_1) = 0$

(10)

By redefining the time variable as $\tau_1 \overset{\cdot}{=} t_1 - t$, a pure initial value problem with $V(x, \tau_1 = 0) = 0$ is obtained. The Cauchy-Kowalewski Theorem [9] can be applied to guarantee local existence and uniqueness of solutions to (10) about $\tau_1 = 0$; the foregoing hypotheses further guarantee that such a solution can be uniquely continued to $\tau_1 = t_1$ and thus establish the smoothness prerequisite for the functional expansion technique to be introduced in the next section.

In the sequel, we will consider examples where $\lim t_1 = \infty$; in this case a solution of the algebraic equation obtained by setting the left-hand side of (10) to zero can be guaranteed under the additional hypothesis that V satisfies the conditions of Lyapunov's second theorem [10]. Bellman and Bucy [11] explored analytic methods for this problem under the further assumption that $\tilde{f}(x,u) = 0$ in (5); their work indicates the considerable complexity of the nonlinear feedback problem.

Equation (10) may be regarded as a special case of a more general class of problems to which Bellman's functional expansion technique applies; we use it as motivation to introduce the more general notation for the next section. Transferring all terms in V to the left-hand side and identifying v with V, and g with Q, the only remaining term on the right-hand side, we may write

$$N_v = g \quad ; \quad v(x, \tau_1 = 0) \equiv 0 \tag{11}$$

where the nonlinear operator N can be viewed as a mapping from the Banach space $C^{1,2}(R^n \times [0, t_1]; R)$ into itself, or more generally as a mapping from one Banach space to another.[2]

By virtue of the preceding assumptions, there is a related linear-quadratic control problem whose solution is known; the HJB equation for it is denoted

$$N_0 v_0 = g_0 \quad ; \quad v_0(x, \tau_1 = 0) \equiv 0 \tag{12}$$

---

[2]Further definition of the topology is omitted because it would unnecessarily complicate the presentation.

where

$$N_0 v_0 = \frac{\partial v_0}{\partial t} + \frac{1}{4} \frac{\partial v_0'}{\partial x} BB' \frac{\partial v_0}{\partial x} - \frac{\partial v_0'}{\partial x} Ax \quad ; \quad g_0 \overset{\bullet}{=} x'Qx$$

and g_0 is the first non-zero term in the Taylor series expansion of $Q(x)$; $Q \epsilon R^{n^2}$ is a positive-definite symmetric matrix by virtue of assumptions (i)-(iii).

The main point of this brief paper is that eq. (12), the natural auxiliary equation for solving (11) happens to be nonlinear and thus a nonlinear extension of Bellman's technique is required.

III. A Functional Expansion Technique

The objective is to determine the solution of a nonlinear functional equation, $Nv = g$, on the basis of the known solution of a related nonlinear auxiliary equation, $N_0 v_0 = g_0$. This is done by constructing a sequence of approximating solutions v^k, $k = 0,1,\ldots$, with v^0 satisfying the auxiliary equation, such that $\lim_{k \to \infty} v^k = v$. The continuation method [12] for the construction of $\{v^k\}$ is based on interpreting the identity

$$N_0 v = g_0 + [(N_0 - N)v + (g - g_0)]$$ (13)

as a continuation to $\varepsilon = 1$ of the class of problems

$$N_0 v = g_0 + \varepsilon[(N_0 - N)v + (g - g_0)]$$ (14)

Note that when $\varepsilon = 0$, v_0 solves (14) and when $\varepsilon = 1$, the desired solution, v, solves (14). Thus it is natural to expect that there should exist a series expansion in ε for v about v_0:

$$v_\varepsilon \overset{\cdot}{=} v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots$$ (15)

If $||(N_0 - N)v||$ and $||g - g_0||$ are sufficiently small, one may expect that the series will have a radius of convergence greater than unity (the norms being interpreted in an appropriate space). In this case the sequence of approximating solutions

$$v^k = \sum_{i=0}^{k} v_i$$

will converge as desired.

Bellman [13] introduced a "fundamental technique" for recursively computing the function v_{i+1} in terms of $v_0, \ldots, v_i$, which was illustrated by means of an example involving a linear auxiliary equation. We shall illustrate the essential procedure for using a nonlinear auxiliary equation and show that the recursion relation remains <u>linear</u> in v_{i+1}. Formally, the procedure is to substitute the series expression (15) into (14) and to equate terms in like powers of ε. Volterra [14, p. 24] has developed an "extension of Taylor's theorem" to a class of functionals, which can be applied to show that there exist operators N_1, N_2, etc. such that

$$N_0 v_\varepsilon = N_0(v_0 + \varepsilon v_1 + \ldots) \tag{16}$$

$$= N_0 v_0 + \varepsilon N_1(v_0)v_1 + \varepsilon^2 N_2(v_0, v_1)v_2 + \ldots$$

and operators ΔN_1, ΔN_2, etc., such that

$$\Delta N_0 v_\varepsilon \overset{\bullet}{=} (N_0 - N)v_\varepsilon$$

$$= (N_0 - N)(v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots)$$

$$= \Delta N_0 v_0 + \varepsilon \Delta N_1(v_0)v_1 + \varepsilon^2 \Delta N_2(v_0, v_1)v_2 + \ldots \tag{17}$$

A further property of the operators N_i, $i \geq 1$ is asserted by the following lemma:

Lemma: Under the conditions described in [14, p. 24], the operators N_i are affine in v_i, given $v_0 \cdots v_{i-1}$, i.e.

$$N_i(v_0, \ldots v_{i-1})v_i = L(v_0)v_i + \tilde{N}_i(v_0, \ldots, v_{i-1}) \qquad (18)$$

This result is established in the appendix.

From (14)-(18) we may then conclude that the v_i can be computed recursively from the equations

$$N_0 v_0 = g_0 \quad \text{(solution assumed known)}.$$

$$L(v_0)v_1 = \Delta N_0 v_0 - \Delta g_0 - \tilde{N}_1(v_0) ; \quad \Delta g_0 = g_0 - g$$

$$L(v_0)v_2 = \Delta N_1(v_0)v_1 - \tilde{N}_2(v_0, v_1) \qquad (19)$$

$$\ldots$$

$$L(v_0)v_k = \Delta N_{k-1}(v_0, \ldots, v_{k-2})v_{k-1} - \tilde{N}_k(v_0, \ldots, v_{k-1})$$

We remark that for the case of a linear auxiliary equation (as considered by Bellman), say $N_0 = L_0$, then the operators L in (19) are independent of v_0, and are in fact all equal to L_0. In the special case noted previously, Bellman and Bucy [11] obtained a procedure similar to (19).

To illustrate the recursion (19), we apply it to the particular problem (10) with the simplifying assumption $F_{2i} = 0$, $i = 1, \ldots, n$, and show that more can be said. The first equation of (19) corresponds to (12). By applying the Lemma, it can be shown that the n-th equation takes the form

$$L(V_0)V_k = \frac{\partial V_k}{\partial t} - \left[\frac{1}{4} \frac{\partial V_k'}{\partial x} BB' \frac{\partial V_0}{\partial x} + \frac{1}{4} \frac{\partial V_0'}{\partial x} BB' \frac{\partial V_k}{\partial x} \right] + \frac{\partial V_k'}{\partial x} Ax$$

$$= \Delta N_{k-1}(V_0, \ldots, V_{k-2})V_{k-1} - \tilde{N}_k(V_0, \ldots, V_{k-1}) \tag{20}$$

It is well-established ([1], [5], [15]) that (12) has the solution

$$V_0(x, t) = x' K_0(t)x \tag{21}$$

with

$$- \frac{d}{dt_1} K_0 = K_0 A + A' K_0 - K_0 BB' K_0 + Q; \quad K(t_1) = 0 \tag{22}$$

and thus

$$[L(V_0)V_k](x,t) = \frac{\partial V_k}{\partial t}(x,t) + x'(A' - K_0(t)BB') \frac{\partial V_k}{\partial x}(x,t); \quad V_k(x,t_1) \equiv 0 \tag{23}$$

To find $V_k(x,t)$ then, requires the solution of a linear variable-coefficient first-order partial differential equation in $n+1$ independent variables.

While a complete convergence analysis lies beyond the scope of this paper, we merely indicate some of the considerations involved. Since $V_k : R^n \times [t_0, t_1] \to R$, some means of dealing with the unboundedness of $\|x\|$ must be provided if numerical solutions are to be considered. One means is to seek a solution for $\|x\| \leq \rho$, where ρ is chosen sufficiently large to accomodate all initial conditions of interest; in this case it is necessary to verify that the solution for $\|x\| \leq \rho$ at time t does not depend on the solution for $\|x\| \geq \rho$, for $t \epsilon [t, t_1]$. That this is in fact the case, can be seen by

-13-

applying the method of characteristics to (20), and noting from (23) that
the hypothesis (iv) of complete controllability of (A,B), along with hypo-
thesis (i)-(ii), which imply that Q is positive definite, imply further that
$(A_0-BB'K_0(t))$ is uniformly stable, and thus the characteristics diverge from
$x=0$ as τ_1 increases from 0 to t_1. Also, it is to be expected that
$\lim\limits_{\|x\| \to \infty} V_k(x,t) = \infty$ for each $t\epsilon[t_0,t_1]$; hence the best convergence result

that might be expected is that $\|V(x,t)\| \leq Me^{\alpha\|x\|}$ for some values of $M>0$
and $\alpha>0$ this is much weaker that the usual bounds employed in solving p.d.e.'s
Finally, the aforementioned conditions on $\|(N_0-N)v\|$ and $\|g-g_0\|$ relate
to the rate of growth of the driving terms on the right-hand side of (20)
as n increases; in fact, these terms have been approximated to third order
by the proposed auxiliary equation.

A useful technique that is exploited extensively in the examples of
the following section is separation of variables. Certain problems may be
solved exactly by this method. Even in cases of numerical computation, a
multinomial expansion of V_n can reduce the computations to solving a
finite system of ordinary differential equations in the time-variable alone.
In these cases, the control law approximation

$$u^k(t) = - \frac{1}{2} \left\{ B' \frac{\partial V^k}{\partial x}(x^k(t),t) + \sum_{i=1}^{n} \frac{\partial V^k}{\partial x_i}(x^k(t),t)F_{1i}f_1(x^k(t)) \right\}$$

$$= - \frac{1}{2} \sum_{j=1}^{k} \left\{ B' \frac{\partial V_j}{\partial x}(x^k(t),t) + \sum_{i=1}^{n} \frac{\partial V_j}{\partial x_i}(x^k(t),t)F_{1i}f_1(x^k(t)) \right\}$$

(24)

can also be expressed as a power series in x^k (the solution of (5) with control u^k); however, this series is not in general the same as the power series for the optimal control law obtained directly from a series solution of (10).

Although there is a formal correspondence in these cases between solutions obtained by power series and by the functional expansion technique it is thus difficult to evaluate the relative computational merits of the two procedures. The relative difficulty of computing the functional expansion operators ΔN_i, $\tilde{N}_i$, $i \geq 1$ must be balanced by the relative simplicity of solving the recursion (19), for which standard numerical procedure are available, and the relative ease of testing for convergence. Further numerical analysis of the functional expansion technique appears warranted, particularly for problems that are not readily amenable to power-series solutions.

IV. Examples

Example 1:

A scalar control u on $[t_0, t_1] = [0, T]$ is sought to minimize

$$J(u) = \int_0^T (u^2(t) + x^2(t) + \frac{1}{2} x^4(t)) dt \tag{25}$$

subject to the scalar state equation

$$x(t) = u(t) + \mu x^3(t) \quad ; \quad x(t_0) = x_0 \tag{26}$$

The optimal control law corresponding to (9) is

$$u^*(t) = -\frac{1}{2} \frac{\partial V}{\partial x} (x^*(t), t) \tag{27}$$

and the HJB equation corresponding to (10) is:

$$\frac{\partial V}{\partial t} = \frac{1}{4} \left(\frac{\partial V}{\partial x} \right)^2 - \mu x^3 \frac{\partial V}{\partial x} - x^2 - \frac{1}{2} x^4 \quad ; V(x, T) = 0 \tag{28}$$

The system (19) takes the form

$$\left.\begin{aligned}
&\frac{\partial V_0}{\partial t} - \frac{1}{4} \left(\frac{\partial V_0}{\partial x} \right)^2 = -x^2 \; ; \; V_0(x, T) = 0 \\[2ex]
&\frac{\partial V_1}{\partial t} - \frac{1}{2} \left(\frac{\partial V_0}{\partial x} \right) \frac{\partial V_1}{\partial x} = -\frac{x^4}{} - \mu x^3 \frac{\partial V_0}{\partial x} \; ; \; V_1(x, T) = 0 \\[2ex]
&\qquad\qquad \cdots \\[1ex]
&\frac{\partial V_k}{\partial t} - \frac{1}{2} \left(\frac{\partial V_0}{\partial x} \right) \frac{\partial V_k}{\partial x} = \frac{1}{4} \left[\sum_{i=1}^{k-1} \left(\frac{\partial V_{k-i}}{\partial x} \right) \left(\frac{\partial V_i}{\partial x} \right) \right] - \mu x^3 \frac{\partial V_{k-i}}{\partial x} \; ; \; V_k(x, T) = 0
\end{aligned}\right\} \tag{29}$$

$$k = 2, 3, \ldots$$

The solution of the first (auziliary) equation is known [17] to be

$$V_0(x,t) = K_0(t)x^2 \; ; \; \frac{dK_0(t)}{dt} - K_0^2(t) + 1 = 0 \; ; \; K_0(T) = 0 \qquad (30)$$

so $K_0(t) = \tanh(T-t)$. $V_1(x,t)$ may be found by separation of variables as

$$V_1(x,t) = K_1(t)x^4; \; \frac{dK_1(t)}{dt} - 4\tanh(T-t)K_1(t) = -\frac{1}{2} + 2\mu\tanh(T-t); \; K_1(T) = 0$$

$$(31)$$

which has the solution

$$K_1(t) = \frac{\mu}{2}[1 - \cosh^{-4}(T-t)] + [\frac{3}{16}(T-t) + \frac{1}{8}\sinh 2(T-t)$$

$$+ \frac{1}{64}\sinh 4(T-t)]\cosh^{-4}(T-t) \qquad (32)$$

Similarly,

$$V_2(x,t) = K_2(t)x^6; \; \frac{dK_2(t)}{dt} - 6K_0(t)K_2(t) = 4K_1^2(t) - 4\mu K_1(t); \; K_2(T) = 0$$

$$(33)$$

All of the succeeding equations may be solved by separation of variables and use of the known variation of constants formula for the solution of a scalar time-varying linear equation. The optimal control approximation thus assumes the form

$$u^k(t) = -\frac{1}{2} \sum_{i=0}^{k} (i+1)K_i(t)x^{(2i+1)} \qquad (34)$$

It should be noted that this is not the same type of approximation obtained from ordinary power series or from singular perturbations in the parameter μ (if its value is small).

Example 2:

A scalar bilinear control problem on $[t_0, t_1] = [0, \ ]$ is to minimize

$$J(u) = \int_0^\infty (u^2(t) + x^2(t) + \frac{1}{2} x^4(t) + x^6(t)) dt \tag{35}$$

subject to

$$\dot{x}(t) = ax(t) + bu(t) + cu(t)x(t); \ x(0) = x_0 \tag{36}$$

The optimal control law corresponding to (9) is

$$u^*(t) = -\left(\frac{b+cx^*(t)}{2}\right) \frac{\partial V}{\partial x}(x^*(t), t) \tag{37}$$

The limiting form of the HJB equation (as $t_1 \to \infty$) is

$$0 = \frac{b}{4}\left(\frac{\partial V}{\partial x}\right)^2 - ax \frac{\partial V}{\partial x} - x^2 - \left[\frac{cx}{4}\left(\frac{\partial V}{\partial x}\right)^2 + \frac{x^4}{2} + x^6\right] \tag{38}$$

The limiting form of (19) is:

$$\frac{b}{4}\left(\frac{\partial V_0}{\partial x}\right)^2 - ax\left(\frac{\partial V_0}{\partial x}\right) = x^2$$

$$\left(\frac{b}{2}\frac{\partial V_0}{x} - ax\right)\left(\frac{\partial V_1}{\partial x}\right) = \frac{c}{4} x \left(\frac{\partial V_0}{\partial x}\right)^2$$

$$\left(\frac{b}{2}\frac{\partial V_0}{\partial x} - ax\right)\left(\frac{\partial V_2}{\partial x}\right) = \frac{c}{2} x \left(\frac{\partial V_0}{\partial x}\right)\left(\frac{\partial V_1}{\partial x}\right) + \frac{x^4}{2} - \frac{b}{4}\left(\frac{\partial V_1}{\partial x}\right)^2$$

$$\left(\frac{b}{2} \frac{\partial V_0}{\partial x} - ax\right)\frac{\partial V_3}{\partial x} = \frac{c}{4} x \left(\sum_{i=0}^{2} \frac{\partial V_{2-i}}{\partial x} \frac{\partial V_i}{\partial x}\right) - \left(\sum_{j=1}^{2} \frac{\partial V_{2-j}}{\partial x} \frac{\partial V_j}{\partial x}\right)$$

$$\left(\frac{b}{2} \frac{\partial V_0}{\partial x} - ax\right)\frac{\partial V_4}{\partial x} = \frac{c}{4} x \left(\sum_{i=0}^{3} \frac{\partial V_{3-i}}{\partial x} \frac{\partial V_i}{\partial x}\right) + x^6 - \left(\sum_{j=1}^{3} \frac{\partial V_{3-j}}{\partial x} \frac{\partial V_j}{\partial x}\right)$$

$$. . .$$

$$\frac{b}{2}\left(\frac{\partial V_0}{\partial x} - ax\right)\frac{\partial V_k}{\partial x} = \frac{c}{4} x \left(\sum_{i=0}^{k-1} \frac{\partial V_{k-i-1}}{\partial x} \frac{\partial V_i}{\partial x}\right) - \left(\sum_{j=1}^{k-1} \frac{\partial V_{k-j-1}}{\partial x} \frac{\partial V_j}{\partial x}\right)$$

$$k = 5,6,7 \tag{39}$$

Applying the same idea as in the previous example we find

$$V_0(x) = K_0 x^2 \; ; \; bK_0^2 - 2K_0 a - 1 = 0 \tag{40}$$

which has the (stable) solution $K_0 = \dfrac{a + \sqrt{a^2+b}}{b}$. Taking

$$V_1(x) = K_1 x^3 \; ; \; 3bK_1K_0 - 3K_1 a = cK_0^2 \tag{41}$$

gives $K_1 = c [3bK_0-3a]^{-1} K_0^2$. Similarly, the coefficients for $V_k(x) = K_k x^{k+2}$ can be identified. The approximate optimal control is then

$$u^k(t) = -\left(\frac{b+cx}{2}\right) \sum_{i=0}^{k} (i+2)K_i x^{(i+1)} \tag{42}$$

In this steady-state case where power-law solutions for $V_i(x)$ can be assumed we thus see that the coefficients will be uniquely determined from the solution of linear equations, once the proper (stabilizing) solution of the auxiliary equation is chosen.

V. Conclusions

In this brief paper, we have pursued the extension of a functional expansion technique suggested originally by Bellman and showed that it has interesting consequences when applied to the solution of the Hamilton-Jacobi-Bellman partial differential equation for a certain class of optimal control problems. While this is perhaps implicit in Bellman's own work, we consider it worthwhile to have clarified the nature of the continuation hypothesis involved, to have identified the explicit requirements for convergence (although a formal proof has not been provided), and to have more clearly delineated the class of problems where the technique is potentially most useful. Furthermore, we have distinguished this technique from power series methods which have been more commonly applied, but which can yield inferior solutions to highly nonlinear or time-varying problems.

Appendix: Demonstration of Lemma

Given a nonlinear and sufficiently regular operator N_0 operating on a convergent power series

$$v_\epsilon = \sum_{i=0}^{\infty} v_i \epsilon^i \qquad (43)$$

it is to be demonstrated that the functional expansion

$$N_0 v_\epsilon = \sum_{i=0}^{\infty} N_i(v_0, \ldots v_{i-1}) v_i \, \epsilon^i \qquad (44)$$

has affine terms

$$N_i(v_0, \ldots v_{i-1}) v_i = L(v_0) v_i + \tilde{N}_i(v_0, \ldots, v_{i-1})$$

This can be seen by direct recourse to Volterra's definitions of the operators N_i [14, p.24]:

$$N_i(v_0, \ldots v_{i-1}) v_i = \frac{1}{i!} \left[\frac{d^i}{d\epsilon^i} \right] N_0(v) \bigg|_{\epsilon=0} \qquad i = 1, 2, \ldots \qquad (45)$$

For instance (i=1):

$$N_1(v_0) v_1 = \int_{t_0}^{t_1} \int_{R^n} \frac{\delta N_0(v_0)}{\delta v} [x,t] \, v_1(x,t) \, dx \, dt \qquad (46)$$

where $\delta N_0/\delta v_0$ is the functional (Frechet) derivative of N_0 with respect to v at v_0 and evaluated at $[x,t] \epsilon R^n x [t_0, t_1]$. This is seen to be a linear operation on v_1. For i=2, we find

$$\frac{1}{2!}\left[\frac{d^2}{d\,\epsilon^2}\,N_0(v_\epsilon)\right]_{\epsilon=0} = \int_{t_0}^{t_1}\int_{R^n}\frac{\delta\,N_0}{\delta v}\,(v_0)\,[x,t]v_2(x,t)dxdt$$

$$+\frac{1}{2!}\int_{t_0}^{t_1}\int_{R^n}\left[\int_{t_0}^{t_1}\int_{R^n}\frac{\delta^2 N_0}{\delta_v{}^2}\,(v_0)\,[x,t;\xi,\tau]v_1(x,t)dxdt\right]v_1(\xi,\tau)d\xi d\tau$$

$$\doteq\ L(v_0)v_2 + \tilde{N}_2(v_0,v_1) \tag{47}$$

The result follows by induction.

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